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LETTER TO THE EDITOR

Studies of the spectral dimension of the Vicsek snowflake fractal

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Abstract. By an exact real space renormalization group (RSRG) method, we obtain the spectral dimension of the Vicsek snowflake fractal. The result seems to cast some doubt on the range of validity of the Einstein relation on the deterministic fractal.

Although the Einstein relation has been shown to hold for many self-similar structures [1-3], its range of validity has been frequently discussed [4, 5]. We have no grounds to believe that it still holds for non-elastic interaction and plane vibration. Here we consider the Vicsek snowflake (VS) (or checkerboard) fractal [6, 7] for harmonic analysis [8] to derive its spectral dimension $\tilde{d}_s = 2d_f/\tilde{d}_w$ and check the Einstein relation for this determinative fractal. Until now, there has been no evidence that this relation fails to hold for determinative fractals. The Einstein relation is between the fractal dimension d_f, the random walk dimension $\tilde{d}_w$ and the resistivity exponent $\tilde{t}$:

$$\tilde{d}_w = d_f + \tilde{t}. \tag{1}$$

$\tilde{d}_w$ could be related to the spectral dimension by $\tilde{d}_w = 2\tilde{d}_f/\tilde{d}_s$. The spectral dimension $\tilde{d}_s$ appears as a governing exponent in many dynamic processes on both random fractals (statistically self-similar) and determinative fractals. We solve the exponent d_s for the VS fractal by the position space renormalization group (RG) method to treat the critical dynamics based on the scaling argument of Rammal and Toulous [8]. We start from the set of equations of the perpendicular harmonic vibration model on the VS fractal ($d_f = \ln 5/\ln 3$) as shown in figure 1. Note that its equations of motion are actually

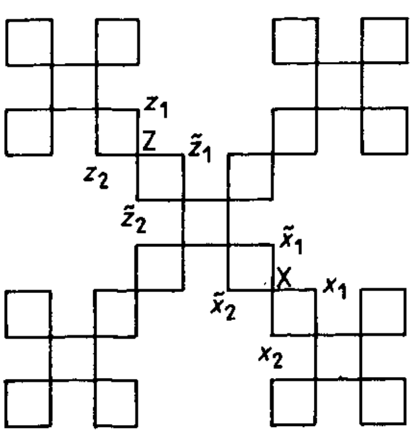

Figure 1. The structure of the VS fractal. The coordinates illustrate the typical RG decimation for reduced frequency λ_i.

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identical to the diluted Heisenberg ferromagnet at zero temperature [9, 10] and similar to the master equation of diffusion. Suppose $\lambda = m\omega^2/k$ and ω, k, m are, respectively, the characteristic frequency, elastic constant and mass. The equation of motion is then as follows:

$$\lambda_n X_i = \sum_j \phi_{i,j}(x_i - x_j) \tag{2}$$

where $\phi_{i,j} = 1$ if i and j are nearest neighbours and $\phi_{i,j} = 0$ in other cases. We have supposed a set of reduced frequencies $\{\lambda_n\}$ corresponding to different characteristic vibrations of various lattices in the procedure of lattice scaling: λ_1 is the reduced frequency for bifurcate points between two quadrafurcate points; λ_2 for bifurcate points between a bifurcate point and a quadrafurcate point; λ_3 for bifurcate points between two bifurcate points; λ_4 for the quadrafurcate points in the generator of the first stage; λ_5 for quadrafurcate points of the second stage (connecting two generators); λ_6 for quadrafurcate points of the third stage, etc.

Referring to figure 1, one may see clearly that the characteristic frequencies for different lattices may be different. We perform RG decimation on equation (2) with a length scaling $b = 3$. The system thus achieves a dilation by the scaling factor b. The outcome of the transformation leads to a set of equations with an exactly identical form to equation (2), but with replacement of new $\{\lambda'_n\}$. Considering the symmetry of the VS fractal, we proceed with, for example, the decimation procedure according to the following channel to approach the recursion relation of λ_4:

$$\lambda_5(X - Z) = 4(X - Z) - [(x_1 + x_2 + \tilde{x}_1 + \tilde{x}_2) - (z_1 + z_2 + \tilde{z}_1 + \tilde{z}_2)] \tag{3a}$$

$$\Downarrow \mathrm{RG}$$

$$\lambda'_4(X - Z) = 4(X - Z) - (X_1 + X_2 - Z_1 - Z_2). \tag{3b}$$

Equation (3b) is the RG-transformed equation. Other relations could be obtained by similar derivations. Our analysis gives the recursion relation of $\{\lambda_n\}$:

$$\begin{aligned}
&\lambda'_1 = \lambda'_2 = 2 - A[B(4 - \lambda_4) - 2(2 - \lambda_3)]/4 + B(2 - \lambda_1) \\
&\lambda'_3 = \lambda_4 - 2 + 2(2 - \lambda_3)/B \\
&\lambda'_4 = \mathscr{F}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = 4 - (4 - \lambda_5)A/B + 4(4 - \lambda_4)/B \\
&\lambda'_5 = \mathscr{F}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_6) \\
&\lambda'_6 = \mathscr{F}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_7) \\
&\ldots
\end{aligned} \tag{4}$$

where $A = (2 - \lambda_1)(4 - \lambda_4) - 2$ and $B = (2 - \lambda_2)(2 - \lambda_3) - 2$.

In order to study the critical behaviour we linearize the above recursion relations for the characteristic frequency around the fixed point $\lambda^* = 0$. One must have noticed that the scaling relations of λ_1 and λ_2 are identical ($\lambda'_1 = \lambda'_2$). This shows that the supposition of different parameters λ_1 and λ_2 is overlapping. But it does not change the maximum eigenvalue of the transforming matrix of $\{\lambda_n\}$. At low frequency, we obtain the transforming matrix $\hat{M}$ of $\{\lambda_n\}$:

$$\hat{M}=\begin{bmatrix} 10 & 5 & 5 & 0 & \dots & \\ 2 & 1 & 1 & 0 & \dots & \\ 4 & -4 & 2 & 3 & 0 & \\ 4 & -4 & 2 & 0 & 3 & \\ 4 & -4 & 2 & 0 & 0 & 3 \\ \vdots & & & \ddots & \dots & \end{bmatrix}. \tag{5}$$

We can solve the maximum eigenvalue $\chi_{\max}$ of $\hat{M}$ by cut-off approximation to give up relations of order $n>l$ in (3). When $l\to\infty$, the computation gives the limit $\chi_{\max}\to 13$. We also find that the matrix reduces to order l when we suppose $\lambda_l=\lambda_{l+1}=\lambda_{l+2}=\dots$ (l can be any arbitrary large number). Thus, we can demonstrate that the value of $\chi_{\max}$, which is independent of l, will be exactly 13. Actually, the transforming matrix will be equivalent to the case of the three RG parameter space $\{\lambda_1(=\lambda_2), \lambda_3, \lambda_4(=\lambda_5=\dots)\}$. Referring to the scaling argument of Rammal and Toulouse [8], one has $\tilde{d}_s=d_f/a$ with $a=\ln(\chi_{\max})^{1/2}$ the scaling exponent [3] of frequency. We obtain the spectral dimension as

$$\tilde{d}_s=\frac{\ln 25}{\ln 13}. \tag{6}$$

We have the order $\tilde{d}_s<d_f<d$, which is same as that of the family of Sierpinski gaskets.

Now let us turn to the Einstein relation $\tilde{d}_w=d_f+\tilde{t}$. $\tilde{d}_w$ is related to the spectral dimension by the Alexander-Orbach relation: $\tilde{d}_w=2d_f/\tilde{d}_s$. The scaling of resistance is rather simple. As indicated in figure 2, we could scale the side resistance t_1 and diagonal resistance t_2. This consideration gives the following recursion equation:

$$\begin{pmatrix} t_1' \\ t_2' \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}. \tag{7}$$

Taking into account the scaling factor $b=3$, we find the scaling exponent of resistance $\tilde{t}=1$.

It is found that the result (6) and value $\tilde{t}$ do not coincide with the relation $2d_f/\tilde{d}_s=d_f+\tilde{t}$. It is generally believed that the Einstein relation is satisfied for the

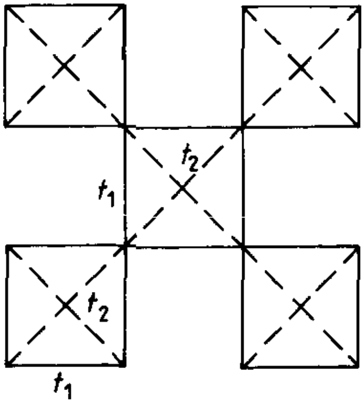

Figure 2. Renormalization of resistance. Diagonal resistances t_2 are denoted by the broken lines.

harmonic vibation model on any fractal lattice. There is even an illustration of the relation for a long-range interaction model by Maritan and Stella [2]. But the VS fractal seems to indicate that it is not generally satisfied. Considering the studies of random walks on Vicsek fractals by Guyer [11], the conclusion on the Einstein relation is contradicting. But, most importantly, the fractal (VS or checkerboard fractal) we discussed is really different from theirs, i.e. the X-type fractals. The spectral dimension of the two-dimensional X-type fractal can be studied [12] by following the same RG procedure; the result agrees with Guyer and does not violate the Einstein relation. So it is the VS fractal that attracts interest. It is distinguished from the X Vicsek fractal in that it has loops. This is more obvious to random walk problems since some loops have only one connecting point with others. We think random walks on this VS fractal with the master equation of [13] will confirm our result.

It should be also pointed out that the overlapping supposition of $\{\lambda_n\}$ does not affect the maximum eigenvalue $\chi_{\max}$. The value of $\chi_{\max}$ is also independent of the sublattice with different mass m_n for the multiatomic fractal (referring to Stinchcombe [10]) or frequency ω_n reflecting in $\lambda_n = m_n\omega_n^2/k$, but depends on the characteristic structure of the lattice which determines the scaling behaviour. One can suppose λ_i to be superfluous, but the difference of characteristic frequencies for various sites must be considered. Here we need at least three RG parameters $\{\lambda_i\}$ as a minimal set to close a direct renormalization.

One may remark that the equations of motion for low-frequency Heisenberg spin wave dynamics [9, 10] are indeed identical to equation (2). By the replacement of $\lambda \to \varpi = \omega/J$, where ω corresponds to the characteristic frequency of the spin wave and J is the nearest-neighbour exchange, and following the same renormalization procedure, one may simply extract the dynamic exponent $z = \ln 13/\ln 3$ for diluted Heisenberg ferromagnetic system at zero temperature on the VS fractal. Thus, we have found an example to prove our supposition of $\{\lambda_n\}$. Bhattacharya [14] studied the critical spin wave dynamics for the Sierpinski gasket-type fractal (STF) and found that his result for the spectral dimension for the STF did not agree with the findings of Hilfer and Blumen [13]. Reconsidering this problem we set different lattices with different characteristic response frequencies as indicated above. The ϖ for the hexafurcate point (ϖ_1) is distinguished from that of quadrafurcate points (ϖ_2). Recalculating the spectral dimension for the STF by the same RG procedure yields $\tilde{d}_s = 2\ln 6/\ln(90/7)$, which is exactly the same result as that of Hilfer and Blumen [13].

In addition, our work is very closely related to that of Ashraff [15]. We propose that three reduced characteristic frequencies are required to make the RG parameter space closed. Ashraff applied the RG technique to Heisenberg spin dynamics on the VS fractal and introduced the next-nearest-neighbour interaction (J_2) because three parameters are required to consummate the recursion relation. However, it is found that the spectral dimension varies with the parameters of the high-nearest-neighbour interaction [2]. The $\tilde{d}_s$ may be changed as J_2 is introduced. From another point of view, the next-nearest-neighbour yields a model that has no loops at all. For random walks the lattice structure is changed and the particle has the possibility of walking diagonally. We suppose that it is the loop which effects the surprising result that the Einstein relation is violated for the VS fractal. However, the fundamental mechanism causing these results needs further study.

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References

[1] Stinchcombe R B 1985 *J. Phys. A: Math. Gen.* **18** L1194
[2] Maritan A and Stella A 1986 *Phys. Rev.* B **34** 456
[3] Tao R 1987 *J. Phys. A: Math. Gen.* **20** 6151
[4] Cates M E 1984 *Phys. Rev. Lett.* **53** 926
[5] Halvin S 1985 *Phys. Rev. Lett.* **55** 130(*c*)
[6] Vicsek T 1983 *J. Phys. A: Math. Gen.* **16** L647
[7] Fraser S J 1988 *Phys. Rev.* A **38** 953
[8] Rammal R and Toulouse G 1983 *J. Phys. Lett.* **44** L13
[9] Harris C K and Stinchcombe R B 1983 *Phys. Rev. Lett.* **50** 1399
[10] Stinchcombe R B and Maggs A C 1986 *J. Phys. A: Math. Gen.* **19** 1949
[11] Guyer R A 1984 *Phys. Rev.* A **30** 1112
[12] Wang Xiaomin *Masters Dissertation* unpublished
[13] Hilfer and Blumen A 1984 *J. Phys. A: Math. Gen.* **17** L537
[14] Bhattacharya S 1985 *J. Phys. A: Math. Gen.* **18** L369
[15] Ashraff R A and Southern B W 1988 *J. Phys. A: Math. Gen.* **21** 2431